Modified Affine Hecke Algebras and Drinfeldians of Type A [1]

V.N. Tolstoy[1], **O.V. Ogievetsky**[2],
P.N. Pyatov[3] and **A.P. Isaev**[3]

[1]*Institute of Nuclear Physics, Moscow State University*
119899 Moscow & Russia
(e-mail: tolstoy@anna19.npi.msu.su)

[2]*Marseille University and Center of Theoretical Physics, CNRS*
Luminy - Case 907-13288 Marseille Cedex 9 & France
(e-mail: oleg@cptsu5.univ-mrs.fr)

[3]*Bogoliubov Laboratory of Theoretical Physics, JINR*
Joint Institute of Nuclear Reserch
141980 Dubna, Moscow region & Russia
(e-mails: pyatov@thsun1.jinr.ru, isaevap@thsun1.jinr.ru)

Abstract

We introduce a modified affine Hecke algebra by a singular transformation of the usual affine Hecke algebra $\hat{H}_q(l)$ of type A_{l-1}. The modified affine Hecke algebra $\hat{H}_{q\eta}(l)$ ($\hat{H}^+_{q\eta}(l)$) depends on two deformation parameters q and η. When the parameter η is equal to zero the algebra $\hat{H}_{q\eta=0}(l)$ coincides with $\hat{H}_q(l)$, if the parameter q goes to 1 the algebra $\hat{H}^+_{q=1\eta}(l)$ is isomorphic to the degenerate affine Hecke algebra $\Lambda_\eta(l)$ introduced by Drinfeld. We construct a functor $\mathcal{F}_{q\eta}$ from a category of representations of $H^+_{q\eta}(l)$ into a category of representations of Drinfeldian $D_{q\eta}(sl(n+1))$ which has been introduced by the first author. This functor depends on two continuous deformation parameters q and η. If the parameter η is equal to zero then the functor $\mathcal{F}_{q\eta=0}$ coincides with the duality functor constructed by Chari and Pressley for the affine Hecke algebra $\hat{H}^+_q(l)$ and the quantum affine algebra $U_q(sl(n+1)[u])$. When the parameter q goes to 1 the functor $\mathcal{F}_{q=1\eta}$ coincides with Drinfeld's functor for the degenerate affine Hecke algebra $\Lambda_\eta(l)$ and the Yangian $Y_\eta(sl(n+1))$.

1 Introduction

One of the most remarkable results of the classical representation theory is the Frobenius-Schur duality between the finite-dimensional irreducible representations of the general or special linear groups and symmetric groups. The duality means that any finite-dimensional irreducible representation of the Lie algebra g (or its universal enveloping algebra $U(g)$), where $g = gl(n+1)$ or $sl(n+1) \simeq A_n$, can be obtained by decomposing of the l-fold tensor product of the fundamental

[1]The talk given by V.N. Tolstoy

(natural) representation $V = \mathbb{C}^{n+1}$ with respect to the action of the symmetric group $S(l)$ (or its group algebra $\mathbb{C}[S(l)]$).

After discovery of the quantum groups [4, 6], Jimbo [7] proved the q-analog of the Frobenius-Schur duality replacing $U(g)$ by $U_q(g)$ and $\mathbb{C}[S(l)]$ by its q-analogue $H_q(l)$, the Hecke algebra of type A_{l-1}. Slightly earlier in 1985, Drinfeld [5] discovered an analogue of the Frobenius-Schur theory for the Yangian $Y_\eta(sl(n+1))$ and the degenerate affine Hecke algebra $\Lambda_\eta(l)$. Later, Chari and Pressley [2] proved the q-analogue of the duality for the quantum affine algebra $U_q(\widehat{sl}(n+1))$ and the affine Hecke algebra $\hat{H}_q(l)$.

In this paper, we extend the results of Drinfeld and Chari-Pressley to the case of the Drinfeldian $D_{q\eta}(sl(n+1))$ [12] - [14] which is the rational-trigonometric deformation of the universal enveloping algebra of the loop algebra $sl(n+1)[u]$. In this case, the role of $\hat{H}_q(l)$ is played the modified affine Hecke algebra $\hat{H}_{q\eta}^{+}(l)$ which we obtain by a singular transformation of the affine Hecke $\hat{H}_q(l)$. Our functor $\mathcal{F}_{q\eta}$ from a category of representations of $H_{q\eta}^{+}(l)$ in a category of those of the Drinfeldian $D_{q\eta}(sl(n+1))$ depends on two continuous deformation parameters q and η. If the parameter η is equal to zero then the functor $\mathcal{F}_{q\eta=0}$ coincides with the duality functor constructed by Chari and Pressley [2] for the affine Hecke algebra $\hat{H}_q^{+}(l)$ and the quantum affine algebra $U_q(sl(n+1)[u])$. When the parameter q goes to 1 the functor $\mathcal{F}_{q=1\eta}$ coincides with Drinfeld's functor for the degenerate affine Hecke algebra $\Lambda_\eta(l)$ and the Yangian $Y_\eta(sl(n+1))$ [5].

2 Affine Hecke and modified affine Hecke algebras

We start from the definition of the affine Hecke algebra [1, 3, 10].

Definition 2.1 *The affine Hecke algebra $\hat{H}_q(l) := \hat{H}_q(A_{l-l})$ of type A_{l-1} is an associative algebra over $\mathbb{C}[q, q^{-1}]$, generated by the elements $\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \ldots, \sigma_{l-1}^{\pm 1}$, and $z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_l^{\pm 1}$ with the following defining relations:*

$$\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1 , \tag{2.1}$$

$$\sigma_i - \sigma_i^{-1} = (q - q^{-1}) , \tag{2.2}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} , \tag{2.3}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if } \ |i-j| > 1 , \tag{2.4}$$

$$z_jz_j^{-1} = z_j^{-1}z_j = 1 , \tag{2.5}$$

$$z_jz_k = z_kz_j , \tag{2.6}$$

$$\sigma_iz_j = z_j\sigma_i \qquad \text{if } \ j \neq i \text{ or } i+1 , \tag{2.7}$$

$$\sigma_iz_i = z_{i+1}\sigma_i^{-1} . \tag{2.8}$$

An associative algebra generated by the elements $\sigma_i^{\pm 1}$, $i \in \{1, 2, \ldots, l-1\}$, with the defining relations (2.1)–(2.4) is called the Hecke algebra $H_q(l) := H_q(A_{l-1})$.

Sometimes it is useful to use the last relation (2.8) in another forms. Namely applying the relation (2.2) one obtains

$$\sigma_iz_i - z_{i+1}\sigma_i = (q^{-1} - q)z_{i+1} \tag{2.9}$$

or

$$z_i\sigma_i - \sigma_i z_{i+1} = (q^{-1} - q) z_{i+1} \ . \tag{2.10}$$

The permutation relations for the inverse powers of the generators z_i looks like

$$\begin{aligned} z_i^{-1}\sigma_i - \sigma_i z_{i+1}^{-1} &= (q^{-1}-q) z_i^{-1} \ , \\ \sigma_i z_i^{-1} - z_{i+1}^{-1}\sigma_i &= (q^{-1}-q) z_i^{-1} \ . \end{aligned} \tag{2.11}$$

Using the relations (2.9)-(2.11) and (2.7) it is easy to see that any polynomial of the elements $\sigma_i^{\pm 1}$ $(i = 1, \dots, l-1)$, and $z_j^{\pm 1}$ $(j = 1, \dots, l)$ may be put in order such that all elements $\sigma_i^{\pm 1}$ are located from the left-hand side (or from the right-hand side) of the elements $z_j^{\pm 1}$, i.e. any polynomials of $\sigma_i^{\pm 1}$ and $z_j^{\pm 1}$ is represented as a sum of the monomials of the type

$$z_1^{n_1} z_2^{n_2} \cdots z_l^{n_l} \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k} \qquad (\text{or} \quad \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k} z_1^{n_1} z_2^{n_2} \cdots z_l^{n_l}) \ , \qquad n_i \in \mathbb{Z}, \tag{2.12}$$

where among the elements σ_{i_j} can be equal. This result is reformulated as the following proposition.

Proposition 2.1 *There is an isomophism of the vector spaces $\hat{H}_q(l)$ and $\mathbb{C}[z_1^{\pm 1}, \dots, z_l^{\pm 1}] \otimes H_q(l)$ (or $H_q(l) \otimes \mathbb{C}[z_1^{\pm 1}, \dots, z_l^{\pm 1}]$):*

$$\hat{H}_q(l) \simeq \mathbb{C}[z_1^{\pm 1}, \dots, z_l^{\pm 1}] \otimes H_q(l) \qquad (\text{or} \ \ \hat{H}_q(l) \simeq H_q(l) \otimes \mathbb{C}[z_1^{\pm 1}, \dots, z_l^{\pm 1}]) \ . \tag{2.13}$$

The subalgebra $\hat{H}_q^+(l) \subset \hat{H}_q(l)$, which is generated by $H_q(l)$ and the elements $z_1, z_2, \dots, z_l$ will be also called the affine Hecke algebra.

The affine Hecke $\hat{H}_q^+(l)$ (and also $\hat{H}_q(l)$) does not contain any singular elements at $q \to 1$ and

$$\lim_{q\to 1} \hat{H}_q(l) \simeq \hat{\Sigma}(l) \ , \qquad \text{and} \qquad \lim_{q\to 1} \hat{H}_q^+(l) \simeq \hat{\Sigma}^+(l) \ , \tag{2.14}$$

where by $\hat{\Sigma}(l)$ $(\hat{\Sigma}^+(l))$ we denote the affine symmetric group algebra generated by the group algebra of the symmetric group $\mathbb{C}[S(l)]$ and the affine elements $z_1^{\pm}, z_2^{\pm}, \dots, z_l^{\pm}$ $(z_1, z_2, \dots, z_l)$ with the defining relation (2.1)-(2.8) for $q = 1$.

Now we introduce a modified the affine Hecke algebra by the singular translation of the affine elements z_j:

$$u_j = z_j + \frac{\eta}{q - q^{-1}} \qquad \text{for } j = 1, 2, \dots, l \ . \tag{2.15}$$

This transformation changes only the last relation (2.8)) from the set (2.1)–(2.8), which takes now the form

$$\sigma_i u_i = u_{i+1}\sigma_i^{-1} + \eta \ . \tag{2.16}$$

A remarkable fact is that while the transformation (2.15) contains terms which are singular, in the classical limit $q \to 1$, the permutation relations (2.16) for the newly defined generators u_i do not. So we have:

Definition 2.2 *The modified affine Hecke algebra $\hat{H}_{q\eta}^+(l) = \hat{H}_{q\eta}^+(A_{l-1})$ of type A_{l-1} is an associative algebra over $\mathbb{C}[q, q^{-1}, \eta]$ generated by the elements $\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \dots, \sigma_{l-1}^{\pm 1}$, and $u_1, u_2, \dots, u_l$*

with the following defining relations:

$$\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1 \;, \tag{2.17}$$

$$\sigma_i - \sigma^{-1} = (q - q^{-1}) \;, \tag{2.18}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \;, \tag{2.19}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if } \; |i-j| > 1 \;, \tag{2.20}$$

$$u_j u_k = u_k u_j \;, \tag{2.21}$$

$$\sigma_i u_j = u_j\sigma_i \qquad \text{if } \; j \neq i \text{ or } i+1 \;, \tag{2.22}$$

$$\sigma_i u_i = u_{i+1}\sigma_i^{-1} + \eta \;. \tag{2.23}$$

The "$\eta - analog$" of the relations (2.9), (2.10) now looks like

$$\begin{aligned} \sigma_i u_i - u_{i+1}\sigma_i &= (q^{-1} - q)u_{i+1} + \eta \;, \\ u_i\sigma_i - \sigma_i u_{i+1} &= (q^{-1} - q)u_{i+1} + \eta \;. \end{aligned} \tag{2.24}$$

It is obvious that the statement of the Proposition 2.1 remains valid for the modified affine Hecke algebra.

One can extend the algebra $\hat{H}^+_{q\eta}(l)$ adding generators u_j^{-1} inverse to the elements u_j: $u_j u_j^{-1} = u_j u_j^{-1} = 1$. In this way one obtains the total modified affine Hecke algebra $\hat{H}_{q\eta}(l)$. However in the present paper we need only the subalgebra $\hat{H}^+_{q\eta}(l) \subset \hat{H}_{q\eta}(l)$.

The algebra $\hat{H}^+_{q\eta}(l)$ is a two-parameter (q,η)-deformation of $\hat{\Sigma}^+(l)$. However it is easy to see that the modified affine Hecke algebra $\hat{H}^+_{q\eta}(l)$ is essentially independent of the parameter η, provided that $\eta \neq 0$. In fact, if $\eta \neq 0$ and $\eta' \neq 0$ the map $\hat{H}^+_{q\eta}(l) \to \hat{H}^+_{q\eta'}(l)$ given by $\sigma_i \mapsto \sigma_i$, $\eta^{-1}u_j \mapsto \eta'^{-1}u_j$ is clearly an isomorphism of these algebras. Thus one might as well take $\eta = 1$, however we keep the parameter η for visualization.

It is obvious that $\hat{H}^+_{q\eta=0}(l) = \hat{H}^+_q(l)$. On the other hand, in the limit $q \to 1$ the modified affine Hecke algebra goes into the degenerate affine Hecke algebra $\Lambda_\eta(l)$ constructed by Drinfeld in 1985 [5] [2]. The relations between the modified affine Hecke algebra $\hat{H}^+_{q\eta}(l)$ and the algebras $\hat{H}^+_q(l)$, $\Lambda_\eta(l)$, $\hat{\Sigma}^+(l)$ (and also their subalgebras) are shown in the picture:

$$\begin{array}{ccc} H_q(l) \subset \hat{H}^+_{q\eta}(l) & \xrightarrow{\eta\to 0} & \hat{H}^+_q(l) \supset H_q(l) \\ {\scriptstyle q\to 1}\big\downarrow & & \big\downarrow{\scriptstyle q\to 1} \\ \Sigma(l) \subset \Lambda_\eta(l) & \xrightarrow{\eta\to 0} & \hat{\Sigma}^+(l) \supset \Sigma(l) \;. \end{array} \tag{2.25}$$

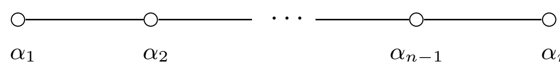
Fig.1. A diagram of the limit algebras of the modified affine Hecke algebra $\hat{H}^+_{q\eta}(l)$ and their subalgebras. The arrows show passages to the limits.

[2]This algebra was also obtained by Drinfeld from the affine Hecke algebra $\hat{H}^+_q(l)$ by letting $q \to 1$ in a certain non-trivial fashion.

3 Drinfeldian and Yangian of type A_n

First we recall the defining relations of the q-quantized universal enveloping algebra $U_q(sl(n{+}1))$ ($sl(n{+}1) := sl(n{+}1, \mathbb{C}) \simeq A_n$) and construction of its Cartan-Weyl basis.

Let $\Pi := \{\alpha_1, \ldots, \alpha_n\}$ be a system of simple roots of $sl(n+1)$ endowed with the following scalar product: $(\alpha_i, \alpha_j) = (\alpha_j, \alpha_i)$, $(\alpha_i, \alpha_i) = 2$, $(\alpha_i, \alpha_{i+1}) = -1$, $(\alpha_i, \alpha_j) = 0$ $((|i-j| > 1)$. The corresponding Dynkin diagram is presented on the picture:



(3.1)

Fig.3. Dynkin diagram of the Lie algebra $sl(n+1)$.

The quantum algebra $U_q(sl(n{+}1))$ is generated by the Chevalley elements $q^{\pm h_{\alpha_i}}$, $e_{\pm\alpha_i}$ $(i = 1, 2, \ldots, n)$ with the defining relations:

$$
\begin{aligned}
q^{h_{\alpha_i}} q^{-h_{\alpha_i}} &= q^{-h_{\alpha_i}} q^{h_{\alpha_i}} = 1 \ , \\
q^{h_{\alpha_i}} q^{h_{\alpha_j}} &= q^{h_{\alpha_j}} q^{h_{\alpha_i}} \ , \\
q^{h_{\alpha_i}} e_{\pm\alpha_j} q^{-h_{\alpha_i}} &= q^{\pm(\alpha_i,\alpha_j)} e_{\pm\alpha_j} \ , \\
[e_{\alpha_i}, e_{-\alpha_j}] &= \delta_{ij}\, [h_{\alpha_i}]_q \\
[e_{\pm\alpha_i}, e_{\pm\alpha_j}] &= 0 \qquad (|i-j| \geq 2) \ , \\
[[e_{\pm\alpha_i} e_{\pm\alpha_j}]_q e_{\pm\alpha_j}]_q &= 0 \qquad (|i-j| = 1) \ ,
\end{aligned}
\tag{3.2}
$$

where $[h]_q := (q^h - q^{-h})/(q - q^{-1})$ is standard notation for the "q-number" and $[\cdot\,,\,\cdot]_q$ is the q-commutator:

$$
[e_\beta, e_\gamma]_q := e_\beta e_\gamma - q^{(\beta,\gamma)} e_\gamma e_\beta \ . \tag{3.3}
$$

The Hopf structure on $U_q(sl(n{+}1))$ is given by the following formulas for a comultiplication Δ_q, an antipode S_q, and a co-unit ε_q:

$$
\begin{aligned}
\Delta_q(q^{\pm h_{\alpha_i}}) &= q^{\pm h_{\alpha_i}} \otimes q^{\pm h_{\alpha_i}} \ , \\
\Delta_q(e_{\alpha_i}) &= e_{\alpha_i} \otimes 1 + q^{-h_{\alpha_i}} \otimes e_{\alpha_i} \ , \\
\Delta_q(e_{-\alpha_i}) &= e_{-\alpha_i} \otimes q^{h_{\alpha_i}} + 1 \otimes e_{-\alpha_i} \ ;
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
S_q(q^{\pm h_{\alpha_i}}) &= q^{\mp h_{\alpha_i}} \ , \\
S_q(e_{\alpha_i}) &= -q^{h_{\alpha_i}} e_{\alpha_i} \ , \\
S_q(e_{-\alpha_i}) &= -e_{-\alpha_i} q^{-h_{\alpha_i}} \ ;
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
\varepsilon_q(q^{\pm h_{\alpha_i}}) &= 1 \ , \\
\varepsilon_q(e_{\pm\alpha_i}) &= 0 \ .
\end{aligned}
\tag{3.6}
$$

Below we shall also use another basis in the Cartan subalgebra of the Lie algebra $sl(n{+}1)$. Namely we set

$$
\begin{array}{rcl}
e_{11} & = & \frac{1}{n+1}\left(nh_{\alpha_1}+(n{-}1)h_{\alpha_2}+\cdots+2h_{\alpha_{n-1}}+h_{\alpha_n}+N\right)\ , \\
e_{22} & = & \frac{1}{n+1}\left(nh_{\alpha_1}+(n{-}1)h_{\alpha_2}+\cdots+2h_{\alpha_{n-1}}+h_{\alpha_n}+N\right)-h_{\alpha_1}\ , \\
\cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
e_{ii} & = & \frac{1}{n+1}\left(nh_{\alpha_1}+(n{-}1)h_{\alpha_2}+\cdots+2h_{\alpha_{n-1}}+h_{\alpha_n}+N\right)-\sum\limits_{k=1}^{i-1}h_{\alpha_k}\ , \\
\cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
e_{n+1n+1} & = & \frac{1}{n+1}\left(-h_{\alpha_1}-2h_{\alpha_2}-\cdots-(n{-}1)h_{\alpha_{n-1}}-nh_{\alpha_n}+N\right)\ .
\end{array}
\tag{3.7}
$$

Here N is a central element of g (and also of $U_q(g)$), which is equal to 0 for the case $g=sl(n{+}1)$ and $N\neq 0$ for $g=gl(n{+}1)$. It is easy to see that

$$
\begin{array}{rcl}
h_{\alpha_i} & = & e_{ii}-e_{i+1i+1} \qquad (i=1,\ldots,n)\ , \\
N & = & e_{11}+e_{22}+\ldots+e_{n+1n+1}\ .
\end{array}
\tag{3.8}
$$

A dual basis to the elements e_{ii} $(i=1,2,\ldots,n{+}1)$ will be denoted by ϵ_i $(i=1,2,\ldots,n{+}1)$: $\epsilon_i(e_{jj})=(\epsilon_i,\epsilon_j)=\delta_{ij}$. In the terms of ϵ_i the positive root system Δ_+ of $sl(n{+}1)$ is presented as follows

$$
\Delta_+=\{\epsilon_i-\epsilon_j\,|\,1\leq i<j\leq n+1\}\ , \tag{3.9}
$$

where $\epsilon_i-\epsilon_{i+1}$ are the simple roots:

$$
\alpha_i=\epsilon_i-\epsilon_{i+1}\quad (i=1,2,\ldots,n)\ . \tag{3.10}
$$

The root $\theta:=\epsilon_1-\epsilon_{n+1}$ is maximal one:

$$
\theta=\alpha_1+\alpha_2+\ldots+\alpha_n\ . \tag{3.11}
$$

For the root vectors $e_{\epsilon_i-\epsilon_j}$ $(i\neq j)$ the standard notations are also used

$$
e_{ij}:=e_{\epsilon_i-\epsilon_j}\ ,\qquad e_{ji}:=e_{\epsilon_j-\epsilon_i}\qquad (1\leq i<j\leq n+1)\ . \tag{3.12}
$$

In particular, e_{ii+1}, e_{i+1i} are the Chevalley elements: $e_{ii+1}=e_{\alpha_i}$, $e_{i+1i}=e_{-\alpha_i}$ $(i=1,\ldots,n)$.

For construction of the composite root vectors e_{ij} $(j\neq i\pm 1)$ we fix the following normal ordering of the positive root system Δ_+ (see [11, 8])

$$
(\epsilon_1-\epsilon_2),(\epsilon_1-\epsilon_3,\epsilon_2-\epsilon_3),\ldots,(\epsilon_1-\epsilon_i,\ldots,\epsilon_{i-1}-\epsilon_i),\ldots,(\epsilon_1-\epsilon_{n+1},\ldots,\epsilon_n-\epsilon_{n+1})\ . \tag{3.13}
$$

According to with this ordering we set

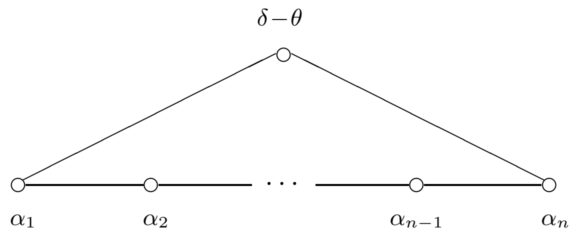
$$
e_{ij}:=[e_{ik},e_{kj}]_{q^{-1}}\ ,\qquad e_{ji}:=[e_{jk},e_{ki}]_q\qquad (1\leq i<k<j\leq n+1)\ . \tag{3.14}
$$

It should be stressed that the structure of the composite root vectors (3.14) is independent of choice of the index k in the r.h.s. of the definition (3.14). In particular one has

$$
\begin{array}{rcl}
e_{ij} & := & [e_{ii+1},e_{i+1j}]_{q^{-1}} = [e_{ij-1},e_{j-1j}]_{q^{-1}} \qquad (1\leq i<j\leq n+1)\ , \\
e_{ji} & := & [e_{ji+1},e_{i+1i}]_q = [e_{jj-1},e_{j-1i}]_q \qquad (1\leq i<j\leq n+1)\ .
\end{array}
\tag{3.15}
$$

General properties of the Cartan-Weyl basis $\{e_{ij}\}$ can be found in [11, 8, 9].

As it was noted in [12] the Dynkin diagrams of the non-twisted affine algebras can be also used for classification of the Drinfeldians and the Yangians. In the case of $sl(n+1)$, the Dynkin diagram of the corresponding affine Lie algebra $\widehat{sl}(n+1)$ is presented by the picture:



(3.16)

Fig.3. Dynkin diagram of the affine Lie algebra $\widehat{sl}(n+1)$.

A general definition of the Drinfeldian $D_{q\eta}(g)$ corresponding to a simple Lie algebra g is given in [12, 13, 14]. The defining relations for generators of $D_{q\eta}(g)$ presented in [12, 13, 14] depend explicitly on the choice of an element $\tilde{e}_{-\theta} \in U_q(g)$ of the weight $-\theta$, such that $g \ni \lim_{q\to 1}\tilde{e}_{-\theta} \neq 0$. Here we present specification of that general definition to the case of $g = sl(n+1)$ and set

$$\tilde{e}_{-\theta} = q^{e_{11}+e_{n+1n+1}} e_{n+11} \ . \tag{3.17}$$

After some calculations we obtain the following result.

Proposition 3.1 *The Drinfeldian $D'_{q\eta}(sl(n+1))$ $(n>1)$ is generated (as a unital associative algebra over $\mathbb{C}[[\log q,\eta]]$) by the algebra $U_q(sl(n+1))$ and the elements $\xi_{\delta-\theta}$, $q^{\pm h_\delta}$ with the relations:*

$$
\begin{aligned}
q^{\pm h_\delta}\text{everything} &= \text{everything}\, q^{\pm h_\delta} \ , && (3.18)\\
q^{e_{11}}\xi_{\delta-\theta} &= q^{-1}\xi_{\delta-\theta}q^{e_{11}} \ , && (3.19)\\
q^{e_{ii}}\xi_{\delta-\theta} &= \xi_{\delta-\theta}q^{e_{ii}} \qquad \text{for } i=2,3,\ldots,n \ , && (3.20)\\
q^{e_{n+1n+1}}\xi_{\delta-\theta} &= q\xi_{\delta-\theta}q^{e_{n+1n+1}} \ , && (3.21)\\
[\xi_{\delta-\theta}, e_{i+1i}] &= 0 \qquad \text{for } i=2,3,\ldots,n-1 \ , && (3.22)\\
[e_{ii+1}, \xi_{\delta-\theta}] &= 0 \qquad \text{for } i=2,3,\ldots,n-1 \ , && (3.23)\\
[e_{12},[e_{12},\xi_{\delta-\theta}]_q]_q &= 0 \ , && (3.24)\\
[[\xi_{\delta-\theta}, e_{nn+1}]_q, e_{nn+1}]_q &= 0 \ , && (3.25)\\
[[e_{12},\xi_{\delta-\theta}]_q,\xi_{\delta-\theta}]_q &= \eta q^{e_{11}+e_{n+1n+1}}\Big(q^{-2}[e_{12},e_{n+11}]\xi_{\delta-\theta} - e_{n+11}[e_{12},\xi_{\delta-\theta}]_q\Big) \ , && (3.26)\\
[[\xi_{\delta-\theta},[\xi_{\delta-\theta},e_{nn+1}]_q]_q &= \eta q^{e_{11}+e_{n+1n+1}+1}\Big(q[e_{n+11},e_{nn+1}]\xi_{\delta-\theta} - e_{n+11}[\xi_{\delta-\theta},e_{nn+1)}]_q\Big) \ . && (3.27)
\end{aligned}
$$

The Hopf structure of $D'_{q\eta}(sl(n+1))$ is defined by the formulas (3.4)-(3.6) for $U_q(sl(n+1))$ (i.e. $\Delta_{q\eta}(x) = \Delta_q(x)$, $S_{q\eta}(x) = S_q(x)$ for $(x \in U_q(g))$) and $\Delta_q(q^{\pm h_\delta}) = q^{\pm h_\delta}\otimes q^{\pm h_\delta}$, $S_q(q^{\pm h_\delta}) = q^{\mp h_\delta}$.

The comultiplication and the antipode of $\xi_{\delta-\theta}$ are given by

$$\begin{aligned}\Delta_{q\eta}(\xi_{\delta-\theta}) &= \xi_{\delta-\theta}\otimes 1+q^{e_{11}-e_{n+1n+1}-h_\delta}\otimes\xi_{\delta-\theta}+\eta\Big(e_{n+11}q^{e_{n+1n+1}}\otimes[e_{11}]\\ &+[\tfrac{h_\delta}{2}+e_{n+1n+1}]q^{-\frac{h_\delta}{2}}\otimes e_{n+11}q^{e_{n+1n+1}}+\sum_{i=2}^{n}e_{n+1i}q^{e_{n+1n+1}}\otimes e_{i1}q^{e_{ii}}\Big)\Big(q^{e_{11}}\otimes q^{e_{11}}\Big),\end{aligned}\tag{3.28}$$

$$\begin{aligned}S_{q\eta}(\xi_{\delta-\theta}) &= -q^{h_\delta-e_{11}+e_{n+1n+1}}\xi_{\delta-\theta}+\eta[\tfrac{h_d}{2}+e_{11}+e_{n+1n+1}+1]q^{\frac{h_\delta}{2}-e_{11}+e_{n+1n+1}-1}e_{n+11}\\ &+\eta\sum_{k=1}^{n}q^{-k}(q-q^{-1})^{k-1}\sum_{n\geq i_k>i_{k-1}>\ldots>i_1\geq 2}e_{n+1i_k}e_{i_ki_{k-1}}\cdots e_{i_11}q^{-2e_{11}}.\end{aligned}\tag{3.29}$$

It is not difficult to check that the substitution $\xi_{\delta-\theta}=q^{e_{11}+e_{n+1n+1}}e_{n+11}$ satisfies the relations (3.18)-(3.27), i.e. there is a simple homomorphism $D_{q\eta}(sl(n+1))\to U_q(sl(n+1))$. Moreover the both sides of the relations (3.26) and (3.27) are equal to zero independently. Therefore we can construct a "evaluation representation" ρ_{ev} of $D_{q\eta}(sl(n+1)$ in $U_q(sl(n+1))\otimes\mathbb{C}[u]$ as follows

$$\begin{array}{rclrcll}\rho_{ev}(q^{h_\delta}) &=& 1\ , & \rho_{ev}(\xi_{\delta-\theta}) &=& uq^{e_{11}+e_{n+1n+1}}e_{n+11}\ , & \\ \rho_{ev}(q^{\pm h_i}) &=& q^{\pm h_i}\ , & \rho_{ev}(e_{\pm\alpha_i}) &=& e_{\pm\alpha_i} & (1\leq i\leq n)\ .\end{array}\tag{3.30}$$

We denote by $D_{q\eta}(sl(n+1))$ the Drinfeldian $D'_{q\eta}(sl(n+1))$ with the central element $h_\delta=0$. It is obvious that

$$D_{q\eta=0}(sl(n+1))\simeq U_q(sl(n+1)[u])\tag{3.31}$$

as Hopf algebras. If $q\to 1$ then the limit Hopf algebra $D_{q=1\eta}(sl(n+1))$ (and also $D'_{q=1\eta}(sl(n+1))$ is isomorphic to the Yangian $Y_\eta(sl(n+1))$ ($Y'_\eta(sl(n+1))$ with $h_\delta\neq 0$) [12]:

$$D_{q=1\eta}(sl(n+1))\simeq Y_\eta(sl(n+1))\ .\tag{3.32}$$

By setting $q=1$ in (3.18)-(3.29), we obtain the defining relations of the Yangian $Y'_\eta(sl(n+1))$ and its Hopf structure in the Chevalley basis. This result is formulated as the proposition.

Proposition 3.2 *The Yangian $Y'_\eta(sl(n+1)$ $(n>1)$ is generated (as an unital associative algebra over $\mathbb{C}[\eta]$) by the algebra $U(sl(n+1))$ and the elements $\xi_{\delta-\theta}$, h_δ with the relations:*

$$[h_\delta,\text{everything}] = 0\ ,\tag{3.33}$$

$$[e_{11},\xi_{\delta-\theta}] = -\xi_{\delta-\theta}\ ,\tag{3.34}$$

$$[e_{n+1n+1},\xi_{\delta-\theta}] = \xi_{\delta-\theta}\ ,\tag{3.35}$$

$$[e_{ii},\xi_{\delta-\theta}] = 0 \qquad \text{for } i=2,3,\ldots,n\ ,\tag{3.36}$$

$$[\xi_{\delta-\theta},e_{i+1i}] = 0 \qquad \text{for } i=2,3,\ldots,n-1\ ,\tag{3.37}$$

$$[e_{ii+1},\xi_{\delta-\theta}] = 0 \qquad \text{for } i=2,3,\ldots,n-1\ ,\tag{3.38}$$

$$[e_{12}[e_{12},\xi_{\delta-\theta}]] = 0\ ,\tag{3.39}$$

$$[[\xi_{\delta-\theta},e_{nn+1}],e_{nn+1}] = 0\ ,\tag{3.40}$$

$$[[e_{12},\xi_{\delta-\theta}],\xi_{\delta-\theta}] = \eta\Big([e_{12},e_{n+11}]\xi_{\delta-\theta}-e_{n+11}[e_{12},\xi_{\delta-\theta}]\Big)\ ,\tag{3.41}$$

$$[[\xi_{\delta-\theta}[\xi_{\delta-\theta},e_{nn+1}]] = \eta\Big([e_{n+11},e_{nn+1}]\xi_{\delta-\theta}-e_{n+11}[\xi_{\delta-\theta},e_{nn+1)}]\Big).\tag{3.42}$$

The Hopf structure of the Yangian is trivial for $U(sl(n+1))\oplus\mathbb{C}h_\delta\subset Y'_\eta(sl(n+1))$ *(i.e.* $\Delta_\eta(x)=x\otimes 1+1\otimes x$, $S_\eta(x)=-x$ *for* $x\in sl(n+1)\oplus\mathbb{C}h_\delta$*) and it is not trivial for the element* $\xi_{\delta-\theta}$*:*

$$\Delta_\eta(\xi_{\delta-\theta}) \;=\; \xi_{\delta-\theta}\otimes 1+1\otimes\xi_{\delta-\theta}+\eta\left(\frac{1}{2}h_\delta\otimes e_{n+11}+\sum_{i=1}^{n+1}e_{n+1i}\otimes e_{i1}\right)\;, \tag{3.43}$$

$$S_\eta(\xi_{\delta-\theta}) \;=\; -\xi_{\delta-\theta}+\eta\Big(\frac{1}{2}h_\delta e_{n+11}+\sum_{i=1}^{n+1}e_{n+1i}e_{i1}\Big)\;. \tag{3.44}$$

An analog of the diagram (2.25) for the Drinfeldian $D_{q\eta}(sl(n{+}1))$ is presented by the picture:

$$\begin{array}{ccc}
U_q(sl(n{+}1))\subset D_{q\eta}(sl(n{+}1)) & \xrightarrow{\eta\to 0} & U_q(sl(n{+}1)[u])\supset U_q(sl(n{+}1)) \\
{\scriptstyle q\to 1}\Big\downarrow & & \Big\downarrow{\scriptstyle q\to 1} \\
U(sl(n{+}1))\subset Y_\eta(sl(n{+}1)) & \xrightarrow{\eta\to 0} & U(sl(n{+}1)[u])\supset U(sl(n{+}1))\;.
\end{array} \tag{3.45}$$

Fig.4. A diagram of the limit Hopf algebras of the Drinfeldian $D_{q\eta}(sl(n{+}1))$ and their subalgebras. The arrows show passages to the limits.

4 Duality between $D_{q\eta}(sl(n{+}1)$ and $\hat{H}^+_{q\eta}(l)$

Let V be the natural $(n+1)$-dimensional representation of the quantum algebra $U_q(sl(n{+}1))$ with basis $\{v_1,v_2,\ldots,v_{n+1}\}$ on which the action of $U_q(sl(n{+}1))$ is given by

$$\begin{aligned}
e_{i-1i}v_k &= \delta_{ik}v_{k+1}\;,\\
e_{i+1i}v_k &= \delta_{ik}v_{k-1}\;,\\
q^{\pm e_{ii}}v_k &= q^{\pm\delta_{ik}}v_k\;.
\end{aligned} \tag{4.1}$$

Let T: $V\otimes V\to V\otimes V$ be a linear map given by

$$T(v_r\otimes v_s)=\begin{cases} qv_r\,\otimes\,v_s & \text{if}\quad r=s\;,\\ \;v_s\,\otimes\,v_r & \text{if}\quad r\leq s\;,\\ \;v_s\,\otimes\,v_r+(q-q^{-1})v_r\otimes v_s & \text{if}\quad r\geq s\;. \end{cases} \tag{4.2}$$

It is not difficult to check that the elements $\sigma_i\in\mathrm{End}_{\mathbb{C}}(V^{\otimes l})$ which act as T on i^{-th} and $(i{+}1)^{-th}$ factors of the tensor product, and as the identity on the other factors, for $i=1,2,\ldots,l$ define the representation of the Hecke algebra $H_q(l)$ on $V^{\otimes l}$.

We say that a representation of $D_{q\eta}(sl(n{+}1)$ has a level l if its restriction to $U_q(sl(n{+}1))$ is sum of representations each of which occurs in $V^{\otimes l}$. Now we announce the main result.

Theorem 4.1 *(i) Let* M *be a finite-dimensional right* $\hat{H}^+_{q\eta}(l)$*-module and we set* $W_M=M\otimes_{H_q(l)}V^{\otimes l}$*. Then there exists a homomorphism* π*:* $D_{q\eta}(sl(n{+}1)\to\mathrm{End}_{\mathbb{C}}W_M$ *such that*

$$\pi(x)(m\otimes\mathbf{v}) \;=\; m\otimes\Delta_q^{(l)}(x)\mathbf{v}\qquad\text{for } x\in U_q(sl(n{+}1))\;, \tag{4.3}$$

$$\pi(\xi_{\delta-\theta})(m\otimes\mathbf{v}) \;=\; m\otimes\Big(\Delta_{q\eta}^{(l)}(\xi_{\delta-\theta})\Big|^{\,i}_{\xi_{\delta-\theta}=u_i}\Big)\mathbf{v} \tag{4.4}$$

for $m \in M$, $\mathbf{v} \in V^{\otimes l}$. For $l \leq n$ the functor $\mathcal{F}_{q\eta}(M)$: $M \to W_M$ is an equivalence between the category of finite-dimensional right $\hat{H}^{+}_{q\eta}(l)$-modules and the category of finite-dimensional left $D_{q\eta}(sl(n{+}1))$-modules of level l.
(ii) For $\eta = 0$ the functor $\mathcal{F}_{q\eta=0}(M)$ is an equivalence between the category of finite-dimensional right $\hat{H}^{+}_{q}(l)$-modules and the category of finite-dimensional left $U_q(sl(n{+}1)$-modules of level $l \leq n$.
(iii) For $q \to 0$ the functor $\mathcal{F}_{q=1\eta}(M)$ is an equivalence between the category of finite-dimensional right $\Lambda_\eta(l)$-modules and the category of finite-dimensional left $Y_\eta(sl(n{+}1)$-modules of level $l \leq n$.

Here $\Delta^{(l)}_{q\eta}$ is the l-fold coproduct

$$\Delta^{(l)}_{q\eta} : D_{q\eta}(sl(n{+}1)) \to D_{q\eta}(sl(n{+}1)) \otimes \cdots \otimes D_{q\eta}(sl(n{+}1)) \qquad (l - \text{fold}) \ . \tag{4.5}$$

In particular

$$\Delta^{(2)}_{q\eta}(\cdot) = \Delta_{q\eta}(\cdot) \tag{4.6}$$

The symbol $\overset{i}{\xi}_{\delta-\theta}= u_i$ in (4.4) means that the i^{-th} component of the affine element $\xi_{\delta-\theta}$ in the l-fold coproduct $\Delta^{(l)}_{q\eta}(\xi_{\delta-\theta})$ has to replace by the affine Hecke element u_i.

The proof of the part (i) of Theorem 4.1 is analogous to the proof of the duality theorem between the affine Hecke algebra $\hat{H}_{q\eta}(l)$ and the quantum affine algebra $U_q(\widehat{sl}(n{+}1))$ (see [2]). The parts (ii) and (iii) are proven by direct comparison of $\mathcal{F}_{q\eta=0}(M)$ and $\mathcal{F}_{q=1\eta}(M)$ with the Chari-Pressley's and Drinfeld's functors [2, 5].

Acknowledgments

The first author (V.N.T.) is grateful to the Org. Committee of the Intern. Symposium "Quantum Theory and Symmetries", H.-D. Doebner, V.K. Dobrev, J.-D. Hennig and W. Lücke, for the support of his visit on this Workshop and he is also thankful to Toulon University for the support of his visit to the Marseille Center of Theoretical Physics CNRS where the first stages of the work were done. This work was supported by the program of French–Russian scientific cooperation (CNRS grant PICS-608 and grant RFBR-98-01-22033) and also by grant RFBR-98-01-00303 (V.N. Tolstoy).

References

[1] Bernshtein I.N. and Zelevinskii A.V., Representation of the group $GL(n, F)$, where F is local Archimedean field, *Usp.Math. Nauk,* **31** (1976), 5-70.

[2] Chari V. and Pressley A., Quantum Affine Algebras and Affine Hecke Algebras, *e-print math.QA/9501003* (1995).

[3] Chari V. and Pressley A., *A guide to quantum groups*, Cambridge University Press, 1994.

[4] Drinfeld V.G., Hopf algebras and quantum Yang-Baxter equation, *Soviet Math. Dokl.,* **283** (1985), 1060-1064.

[5] Drinfeld V.G., Degenerate affine Hecke algebras and Yangians, *Func. Anal. Appl.,* **20** (1986), 62-64.

[6] Jimbo M., A q-difference analogue of $U(g)$ and Yang-Baxter equation, *Lett. Math. Phys.,* **10** (1985), 63-69.

[7] Jimbo M., A q-analogue of $U_q(gl(n+1))$, Hecke algebras and Yang-Baxter equation, *Lett. Math. Phys.,* **11** (1986), 247-252.

[8] Khoroshkin S.M. and Tolstoy V.N., Universal R-matrix for quantized (super)algebras, *Commun. Math. Phys.,***141** (1991), 599-617.

[9] Khoroshkin S.M., and Tolstoy V.N., Twisting of quantum (super)algebras. Connection of Drinfeld's and Cartan-Weyl realizations for quantum affine algebras, *Preprint MPIM Bonn (Germany), MPI/94-23* (1994), 29p.; *e-print hep-th/9404036* (1994).

[10] Rogavski J.D., On modules over the Hecke algebras of p-adic group, *Invent. Math.,* **79** (1985), 443-465.

[11] Tolstoy V.N., Extremal projectors for quantized Kac-Moody superalgebras and some of their applications, *Lectures Notes in Physics 370* (1990), 118-125.

[12] Tolstoy V.N., Connection between Yangians and Quantum Affine Algebras, Proceedings of the X-th Max Born Symposium, (Wroclav, 1996, eds: J. Lukierski, M.Mozrzymas). PWN - Polish Scientific Publishers - Warszawa (1997), 99-117.

[13] Tolstoy V.N., Two-parameter deformations of loop algebras and superalgebras, Proc. of the 5-th Wigner Symposium, (Vienna, 1997, eds: P.Kasperkovitz, D.Grau). World Scientific, Singapore-New Jersey-London-Hong Kong (1998), 25-27; *e-print math.QA/9712028* (1997).

[14] Tolstoy V.N., Drinfeldians, Proc. vol. *"Lie Theory and its Application in Physics II"* (eds: H.-D. Doebner, V.K. Dobrev and J. Hilgert, World Scientific, Singapore, 1998, 981-02-3539-9, pp. 325-337; *e-print math.QA/9803008* (1998).